Basics

A *graph* G is a tuple (V,E) where V is a finite set of vertices and $E \subset \binom{V}{2}$ is the set of edges.

A *multigraph* G is a tuple (V,E) where V is a set of vertices and E is a multiset of elements from $\binom{V}{1} \cup \binom{V}{2}$ i.e. multiedges and loops.

A *hypergraph* H is a tuple (X,E) where X is a finite set and $E \subseteq 2^X \setminus \{\emptyset\}$.
i.e. edges may join any number of vertices.

Notation of special graphs

K_n is the complete graph on n vertices.
C_n is the cycle on n vertices.
P_n is a path of length n where n counts edges.
E_n is the empty i.e. edge-less graph on n vertices.
$K_{m,n}$ is the complete bipartite graph with partite sets of cardinality m respectively n.

Local properties

$N(v)$ for $v \in V(G)$ is the *neighbourhood* of v.
$d(v)$ for $v \in V(G)$ is the *degree* of v.
$d(u,v)$ is the length of a shortest u-v-path.

Global properties

$|G| := |V(G)|$ and $||G|| := |E(G)|$
$\delta(G)$ is the minimum degree in G.
$\Delta(G)$ is the maximum degree in G.

G is k-regular $\iff \forall v \in V(G) : d(v) = k$
The *girth* $g(G)$ is the length of a shortest cycle.
The length of a longest cycle is G's *circumference*.

$\operatorname{diam}(G) := \max_{u,v \in V(G)} d(u,v)$ is the *diameter* of G.

$\operatorname{rad}(G) := \min_{u \in V(G)} \max_{v \in V(G)} d(u,v)$ is the *radius* of G.

Handshake Lemma

$$2|E| = \sum_{v \in V} d(v)$$

Furthermore the sum of all degrees is even and thus #vertices with odd degree is also even.

Induced Subgraphs

Subgraph $H \subset G$ is *induced* if:
$\forall u,v \in V(H) : uv \in E(H) \iff uv \in E(G)$
i.e. every induced subgraph may be generated by deleting vertices and their incident edges from G.

Bipartite Charaterization

G is *bipartite* iff it has no cycles of odd length.

Hamiltonian Cycles

A *Hamiltonian cycle* is a cycle that visits each vertex of G exactly once.
Correspondingly a *Hamiltonian path* is a path that visits each vertex of G exactly once.

Dirac Theorem

Every graph G with $|V(G)| \geq 3$ and $\delta(G) \geq \frac{n}{2}$ has a Hamiltonian cycle.

Eulerian Tour Condition

An *Eulerian tour* is closed walk containing all edges of G exactly once.

A connected graph has an Eulerian tour iff every vertex has even degree.

Hall's Marriage Theorem

Bipartite G with partite sets A, B has a matching containing A iff $\forall S \subset A : |N(S)| \geq |S|$.

Tutte's Theorem

Let $q(G)$ be the number of odd components in G.
G has perfect matching iff $\forall S \subseteq V : q(G-S) \leq |S|$.

Any k-regular bipartite graph has a perfect matching and a proper k-edge coloring.

König's Theorem

For bipartite G the size of the largest matching equals the size of a smallest vertex cover.

Hajnal and Szemerédi

Let H, G be graphs. An *H-factor* of G is a spanning subgraph that is a set of disjoint copies of H in G whose vertex sets form a partition of $V(G)$.

$\delta(G) \geq (1 - \frac{1}{k})n$ for $k|n \implies G$ has a K_k-factor.

Connectivity

Let $\kappa(G)$ be the *connectivity* of G
i.e. the maximum k for which G is *k-connected*.
G is k-connected if $k-1$ vertices can be removed without disconnecting.
Let $\kappa'(G)$ be the edge-connectivity of G:

$$\kappa(G) \leq \kappa'(G) \leq \delta(G)$$

Menger's Theorem

For $A, B \subseteq V(G)$ the min #vertices separating A, B equals the max number of disjoint A-B-paths.
For $a, b \in V(G)$ s.t. $\{a,b\} \notin E(G)$ the minimum number of $v \in V(G) \setminus \{a,b\}$ separating a, b equals the maximum number of independent a-b-paths.

Global Menger's Theorem

Graph G is k-connected iff $\forall a,b \in V(G)$ there exist k independent a-b-paths.

Ear-decomposition

Ear-decomposition of G is a sequence of graphs $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k$ s.t. G_0 is a cycle, G_i results from G_{i-1} by adding an ear and $G_k = G$.
G is 2-connected iff it has an ear-decomposition.

Block-cut-vertex-graph

Blocks of G are the maximal 2-connected subgraphs and bridges.
The *block-cut-vertex-graph* of G is the bipartite graph s.t. its partite sets are the blocks on the one side and the cut-vertices on the other side.
Block-cut-vertex-graph of connected G is a tree.

Planar Graphs

Let a *plane graph* be a set of points on the plane connected by arcs s.t. the arcs do not contain any of the points or intersect any other arc.
A *planar embedding* is a bijection between a plane graph and a abstract graph. A *planar graph* is a graph G with a planar embedding. The corresponding plane graph is a drawing of G.

Fáry's Theorem

Every planar graph can be embedded in a plane s.t. the edges are straight lines.

Euler's Formula

For connected planar G with v vertices, e edges and f faces: $v - e + f = 2$

$|E(G)| \leq 3|V(G)| - 6$ (equal if G is triangulation)
$|E(G)| \leq 2|V(G)| - 4$ if no face is bound by triangle.

Tutte's Theorem

Every 4-connected planar graph is Hamiltonian.

Graph minors

H is *minor* of G (i.e. $MH \subseteq G$) if it can be generated from G by deleting vertices, deleting or contracting edges.
H is a *subdivision* of G if it can be generated from G by subdividing edges.
H is a *topological minor* of G (i.e. $TH \subseteq G$) if a subgraph of G is a subdivision of H.

Kuratowski's Theorem

For graph G the following is equivalent:

(a) G is planar
(b) $K_5, K_{3,3}$ aren't minors of G
(c) K_5, $K_{3,3}$ aren't topological minors of G

Colorings

The *chromatic number* $\chi(G)$ and the *chromatic index* $\chi'(G)$ are defined as follows:

$$\chi(G) := \min_{k \in \mathbb{N}} : G \text{ has proper } k\text{-vertex coloring}$$
$$\chi'(G) := \min_{k \in \mathbb{N}} : G \text{ has proper } k\text{-edge coloring}$$

k-regular bipartite G has proper k-edge-coloring.

Cliques

A *clique* is a subgraph of G that is a complete graph. The *clique number* $\omega(G)$ is the maximum order of a clique in G. The *co-clique number* $\alpha(G)$ is the largest order of an independent set in G.

G is *perfect* if $\forall$ induced $H \subseteq G : \chi(H) = \omega(H)$.
Bipartite graphs are perfect with $\chi = \omega = 2$.

5-Color Theorem

Every planar graph is 5-colorable.

List coloring

$\forall v \in V(G)$ let $L(v) \subseteq \mathbb{N}$ be a list of colors.
G is *L-list-colorable* if $\exists$ a proper coloring c s.t. $\forall v \in V(G) : c(v) \in L(v)$.

G is *k-list-colorable* or *k-choosable* if it is L-list-colorable for all lists of k colors.

Choosability $ch(G) := \min_{k \in \mathbb{N}} \{G \text{ is } k\text{-choosable}\}$

Thomassen's 5-List-Color Theorem

Every planar graph is 5-choosable.

Greedy chromatic number estimate

$$\chi(G) \leq \Delta(G) + 1$$

Brook's Theorem

If G is connected and neither complete nor an odd cycle then $\chi(G) \leq \Delta(G)$

König's Theorem

$$G \text{ bipartite} \implies \chi'(G) = \Delta(G)$$

Vizing's Theorem

$$\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$$

Extremal Graph Theory

For $n \in \mathbb{N}$ and graph H the *extremal number* $ex(n,H)$ is the max number of edges in a graph of order n s.t. it doesn't contain subgraph H.

$$ex(n,H) := \max\{\|G\| : |G| = n, H \not\subseteq G\}$$

Correspondingly $EX(n,H$ is the set of graphs on n vertices and $ex(n,H)$ edges that are H-free.

e.g. $ex(n,K_2) = 0,\ EX(n,K_2) = \{E_n\},\ ex(n,P_3) = \lfloor \frac{n}{2} \rfloor$

Turán Graph

For $1 \le r \le n$ the *Turán graph* $T(n,r)$ is the unique complete r-partite graph of order n s.t. the partite set sizes differ by at most one.

i.e. $T(n,r)$ has n mod r partite sets of size $\lceil \frac{n}{r} \rceil$ and $r - (n \bmod r)$ partite sets of size $\lfloor \frac{n}{r} \rfloor$.
$\forall v \in V(T(n,r)) : d(v) \in \{n - \lceil \frac{n}{r} \rceil, n - \lfloor \frac{n}{r} \rfloor\}$

$T(n,r)$ doesn't contain K_{r+1} and $t(n,r) := \|T(n,r)\|$.
For $r|n$ $T(n,r)$ is denoted by K_r^s where $n = r \cdot s$.

$$t(n,r) = t(n-r,r) + (n-r)(r-1) + \binom{r}{2}$$

Comparing the number of edges in K_n and $T(n,r)$:

$$\lim_{n\to\infty} \frac{t(n,r)}{\binom{n}{2}} = \left(1 - \frac{1}{r}\right)$$

Turán's Theorem

$$EX(n,K_r) = \{T(n,r-1)\}$$

ϵ-regularity

Let $X,Y \subseteq V(G)$ be disjoint and $\|X,Y\|$ is the number of edges between X and Y.

$$d(X,Y) := \frac{\|X,Y\|}{|X||Y|} \text{ is the density of } (X,Y)$$

$\forall \epsilon > 0 : (X,Y)$ is an *ϵ-regular pair* if:
$\forall A \subseteq X, B \subseteq Y :$
$|A| \ge \epsilon|X| \wedge |B| \ge \epsilon|Y| \implies |d(X,Y) - d(A,B)| \le \epsilon$.

$V = V_0 \dot\cup \cdots \dot\cup V_k$ is a *ϵ-regular partition* if:

1. $|V_0| \le \epsilon|V|$
2. $|V_1| = \cdots = |V_k|$
3. At most ϵk^2 pairs (V_i, V_j) for $1 \le i < j \le k$ are not ϵ-regular

Szemerédi's Regularity Lemma

$\forall \epsilon > 0, m \ge 1 \exists M \in \mathbb{N}$ s.t. every graph G with $|G| \ge m$ has ϵ-regular partition $V_0 \dot\cup \cdots \dot\cup V_k$ with $m \le k \le M$.

Erdős-Stone

$\forall r > s \ge 1, \epsilon > 0 \exists n_0 \in \mathbb{N}$ s.t. all graphs with $|V(G)| =: n \ge n_0$ vertices and

$$|E(G)| \ge t_{r-1}(n) + \epsilon n^2$$

contain K_r^s as a subgraph.

Asymptotic extremal number

$$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

e.g. $ex(n, K_5 \setminus \{e\}) \simeq \frac{2}{3} \cdot \binom{n}{2}$ as $\chi(K_5 \setminus \{e\}) = 4$.

Zarankiewicz function

$z(m,n;s,t)$ denotes the maximum number of edges that a bipartite graph on sets of size m and n can have without containing $K_{s,t}$.

Kővári-Sós-Turán Theorem

$z(m,n;s,t) \le (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m$
i.e. for $m = n$ and $t = s$:

$$z(n,n;t,t) \in \mathcal{O}(n^{2-\frac{1}{t}})$$

Ramsey Theory

For $k \in \mathbb{N}$ the *Ramsey number* $R(k) \in \mathbb{N}$ is the min n s.t. every 2-edge-coloring of K_n contains monochromatic K_k.

For $k,l \in \mathbb{N}$ the *asymmetric Ramsey number* $R(k,l)$ is the min n s.t. every 2-edge-coloring of K_n contains red K_k or blue K_l.

For graphs G,H the *graph Ramsey number* $R(G,H)$ is the min n s.t. every 2-edge-coloring of K_n contains red G or blue H.

For graphs G,H the *induced Ramsey number* $R_{\text{ind}}(G,H)$ is the min n s.t. there is graph F on n vertices every 2-coloring of which contains red G or blue H.

For $r, l_1, \ldots, l_k \in \mathbb{N}$ the *hypergraph Ramsey number* $R_r(l_1,\ldots,l_k)$ is the min n s.t. for every k-coloring of $\binom{[n]}{r}$ there $\exists i \in [k], V \subseteq [n]$ with $|V| = l_i$ s.t. all sets in $\binom{V}{r}$ have color i.

e.g. $R(2,k) = R(k,2) = k$ and $R(3) = 6$

Ramsey Theorem

$$\forall k \in \mathbb{N} : \sqrt{2}^k \le R(k) \le 4^k$$

Particularly Ramsey, asymmetric Ramsey and graph Ramsey numbers are finite.

Induced Ramsey Theorem

$R_{\text{ind}}(G,H)$ is finite for all graphs G,H.

Erdős-Szekeres' Theorem

Any sequence of $(r-1)(s-1)+1$ distinct numbers in $\mathbb{R}$ contains an ascending subsequence of length r or a descending subsequence of length s.

Schur's Theorem

Let $\mathbb{N}$ be colored with $r \in \mathbb{N}$ colors.
$\implies \exists x,y,z \in \mathbb{N}$ of the same color s.t. $x + y = z$.

Flows

Let G be a multigraph, $T := \{(x,e,y) | xy = e \in E(G)\}$.
For abelian group H the function $f : T \to H$ defines a *circulation* on G if:

1. $\forall (x,e,y) \in T, x \ne y : f(x,e,y) = -f(y,e,x)$
2. $\forall v \in V(G) : f(v,V) = 0$

Let $f(X,Y) := \sum_{(x,e,y)\in T, x\ne y, x\in X, y\in Y} f(x,y)$

For any circulation f and $X \subseteq V(G)$:
$f(X,X) = 0,\ f(X,V(G)) = 0,\ f(X, V(G) \setminus X) = 0$.

k-flows

For $k \in \mathbb{N}$ f is a *k-flow* if it is a $\mathbb{Z}$-flow s.t. $\forall (x,e,y) \in T : 0 < |f(x,e,y)| < k$.

The smallest k s.t. that G has a k-flow $\varphi(G)$ is the *flow number* of G.

Networks

Let $s \ne t \in V(G)$ and $c : T \to \mathbb{Z}_{\ge 0}$.
Then (G,s,t,c) is a *network* with *source* s, *sink* t and *capacity function* c.

$f : T \to \mathbb{R}$ is a *network flow* if:

1. $f(x,e,y) = -f(y,e,x)$ for all $e \in E(G)$ with endpoints $x \ne y$
2. $f(x, V(G)) = 0, x \in V(G) \setminus \{s,t\}$
3. $f(x,e,y) \le c(x,e,y)$ for all $e \in E(G)$ with endpoints $x \ne y$

Network cuts

$(S, \overline{S})$ for $S \subsetneq V(G)$ s.t. $s \in S, t \notin S, \overline{S} = V(G) - S$ is called a *cut* in the network (G,s,t,c).

Its *capacity* is $c(S,\overline{S}) := \sum_{x\in S, y\in\overline{S}, (x,e,y)\in T} c(x,y)$.

For any flow f and cut $(S,\overline{S}) : f(S,\overline{S}) = f(s, V(G))$.

Ford-Fulkerson Theorem

In any network the maximum value of a flow equals the minimum capacity of a cut.
An integral flow f with this max flow value exists.

Random Graphs

$\mathcal{G}(n,p)$ is the probability space on graphs of order n for which the edge-existence is independently decided by fixed probability $p \in [0,1]$.
This is called *Erdős-Rényi model*.

Properties

A *property* $\mathcal{P}$ is a set of graphs.
$G \in \mathcal{G}(n,p_n)$ *almost always* has property $\mathcal{P}$ if:

$$\mathbb{P}(G \in \mathcal{G}(n,p_n) \cap \mathcal{P}) \to 1 (n \to \infty)$$

If p_n is constant it is said that *almost all* $G \in \mathcal{G}(n,p)$ have property $\mathcal{P}$.

Threshold functions

$f : \mathbb{N} \to [0,1]$ is called *threshold function* for $\mathcal{P}$ if:

1. For $\frac{p_n}{f(n)} \to 0 (n \to \infty)$ graphs $G \in \mathcal{G}(n,p_n)$ almost always don't have property $\mathcal{P}$
2. For $\frac{p_n}{f(n)} \to 1 (n \to \infty)$ graphs $G \in \mathcal{G}(n,p_n)$ almost always have property $\mathcal{P}$

Such functions do not exist for all properties $\mathcal{P}$.

Simple probabilities

Let G be a graph on n vertices and m edges.

$$\mathbb{P}(G = \mathcal{G}(n,p)) = p^m (1-p)^{\binom{n}{2} - m}$$

Expected values and indicators

Let $X : \mathcal{G}(n,p) \to \mathbb{N}$ be a random variable.
When X counts e.g. the number of a certain class of subgraphs $\mathbb{E}(X)$ may be calculated by summing indicators over the maximum number N of such subgraphs in $\mathcal{G}(n,p)$ and applying linearity:

$$\mathbb{E}X = \mathbb{E}\left(\sum_{i=1}^N X_i\right) = \sum_{i=1}^N \mathbb{E}X_i = \sum_{i=1}^N \mathbb{P}(X_i = 1)$$

Expected number of k-cycles

$$\mathbb{E}(\#k\text{-cycles in } G \in \mathcal{G}(n,p)) = \frac{n_k}{2k} \cdot p^k$$

for $n_k := n \cdot (n-1) \cdots (n-k+1)$.

Erdős' lower bound on $R(k,k)$

$$R(k.k) \ge 2^{k/2}$$

Erdős-Hajnal

$\forall k \ge 3 \exists$ a graph G s.t. $g(G), \chi(G) > k$.